

Multi-Skyrmions with Vector Mesons

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Abstract

It is known that including vector mesons stabilizes the size of a Skyrmion without the need for a Skyrme term. This paper provides the first results for static multi-Skyrmions in such a theory. The rational map ansatz is used to investigate multi-Skyrmions in a theory which includes the ω vector meson and has no Skyrme term. Bound states with baryon numbers two, three and four are found, which have axial, tetrahedral and cubic symmetries, respectively. The results reveal a qualitative similarity with the standard Skyrme model with a Skyrme term and no vector mesons, suggesting that some features are universal and do not depend on the details of the theory. Setting the pion decay constant and meson masses to their experimental values leaves only a single free parameter in the model. Fixing this parameter, by equating the energy of the baryon number four Skyrmion to the He^4 mass, yields reasonable results for other baryon numbers.

1 Introduction

Skyrmions are topological solitons that model baryons within a nonlinear theory of pions, arising as a low energy effective theory from QCD in the limit of a large number of colours [1]. The standard Skyrme model [2] includes only the pion degrees of freedom and the Lagrangian requires the inclusion of a Skyrme term, which is quartic in derivatives. The role of the Skyrme term is to balance the sigma model contribution and provide a scale for the soliton, as required by Derrick's theorem [3]. The Skyrme term has some drawbacks, for example, it makes the theory non-renormalizable and the classical dynamical field equations have potential instabilities associated with the loss of hyperbolicity [4].

Over twenty years ago it was shown [5] that by generalizing the nonlinear pion theory to include vector mesons, the size of a Skyrmion is stabilized without the need for a Skyrme term. The original work included only the ω meson but later extensions added other vector mesons, for example ρ mesons [6, 7, 8]. These investigations produced promising results and revealed some improvements over the standard Skyrme model, although all these studies were limited to the sector with baryon number one.

The single Skyrmion solution is spherically symmetric and therefore it can be constructed by solving only ordinary differential equations; though even these must be solved numerically. However, multi-Skyrmions are not spherically symmetric and therefore highly nonlinear partial differential equations in three-dimensional space must be solved to study baryon numbers greater than one. The substantial difficulties that need to be overcome mean that even today there are still no results available on static multi-Skyrmions in theories including vector mesons; though it has been demonstrated that a product ansatz allows well-separated Skyrmions to be placed in an attractive channel [9], and recently a Skyrme crystal has been investigated [10]. The purpose of the present paper is to provide the first results on static multi-Skyrmions in theories including vector mesons. Details are presented for baryon numbers from one to four, but the methods described are also applicable to larger baryon numbers.

Substantial progress in both numerical and analytic approaches to standard Skyrmions means that multi-Skyrmions are now fairly well understood in the Skyrme model (for a review see [11]). The approach taken in this paper is to apply some of these techniques, in particular the rational map ansatz [12], to study multi-Skyrmions in a theory without the Skyrme term but including the ω vector meson. Briefly, the rational map ansatz will be applied to provide an approximation to the Skyrme field and the ω field will be computed using an expansion in terms of symmetry adapted spherical harmonics.

Bound states with baryon numbers two, three and four are found, which have axial, tetrahedral and cubic symmetries, respectively. The results reveal a qualitative similarity with the standard Skyrme model, suggesting that some features are universal and do not depend on the details of the theory. Setting the pion decay constant and meson masses to their experimental values leaves only a single free parameter in the model. Fixing this parameter, by equating the energy of the baryon number four Skyrmion to the He^4 mass, yields reasonable results for other baryon numbers.

Finally, recent developments in AdS/QCD have led to renewed interest in baryons as

Skyrmions [13]. In particular, these studies point to the importance of the inclusion of vector mesons and provide a string theory motivation for old ideas of vector meson dominance.

2 Skyrmions and the ω meson

The Skyrme field U takes values in $SU(2)$ and satisfies the boundary condition that $U \rightarrow 1$ as $|\mathbf{x}| \rightarrow \infty$. It is related to the triplet of pion fields $\boldsymbol{\pi}$ through the formula

$$U = \sigma + i\boldsymbol{\pi} \cdot \boldsymbol{\tau}, \quad (2.1)$$

where $\boldsymbol{\tau}$ denotes the triplet of Pauli matrices and $\sigma^2 + \boldsymbol{\pi} \cdot \boldsymbol{\pi} = 1$.

Topological solitons arise because there is a conserved topological current

$$B^\mu = \frac{1}{24\pi^2} \epsilon^{\mu\nu\alpha\beta} \text{Tr}(\partial_\nu U U^\dagger \partial_\alpha U U^\dagger \partial_\beta U U^\dagger), \quad (2.2)$$

and the associated integer topological charge $B = \int B^0 d^3x$ is the soliton number and is identified with baryon number.

The theory for the Skyrme field coupled to the ω vector meson is given by the Lagrangian density [5]

$$\mathcal{L} = \frac{F_\pi^2}{16} \text{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{F_\pi^2 m_\pi^2}{8} \text{Tr}(U - 1) - \frac{1}{4} (\partial_\mu \omega_\nu - \partial_\nu \omega_\mu) (\partial^\mu \omega^\nu - \partial^\nu \omega^\mu) + \frac{m_\omega^2}{2} \omega_\mu \omega^\mu + \beta \omega_\mu B^\mu, \quad (2.3)$$

where $F_\pi = 186 \text{ MeV}$ is the pion decay constant, $m_\pi = 138 \text{ MeV}$ is the pion mass and $m_\omega = 782 \text{ MeV}$ is the ω mass. The constant β can be related to the $\omega \rightarrow 3\pi$ decay rate, however, this is enhanced by the resonance $\omega \rightarrow \rho + \pi$ which is not included in the current theory. Therefore the experimental data only provides an upper bound on β , which is found to be $\beta \leq 25.4$ [5].

It is convenient to remove various constants in the above Lagrangian density by rescaling the ω field as $\omega \mapsto \omega F_\pi$ and using energy and length units of F_π^2/m_ω and $1/m_\omega$ respectively. The Lagrangian density then becomes

$$\mathcal{L} = \frac{1}{16} \text{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{M^2}{8} \text{Tr}(U - 1) - \frac{1}{4} (\partial_\mu \omega_\nu - \partial_\nu \omega_\mu) (\partial^\mu \omega^\nu - \partial^\nu \omega^\mu) + \frac{1}{2} \omega_\mu \omega^\mu + g \omega_\mu B^\mu, \quad (2.4)$$

where $M = m_\pi/m_\omega = 0.176$ and $g = \beta m_\omega/F_\pi$.

Note that the theory (2.4) does not contain a Skyrme term and has only one free parameter, g , once the meson masses are fixed to their experimental values.

Several values for g have been studied and the results are qualitatively similar in all cases. Results will be presented for the two values $g = 96.7$ and $g = 34.7$, which are both consistent with the above upper bound for β . The first value, which is adopted from now on until further notice, is taken from [5], and is obtained by allowing both g and F_π to be free parameters whose values are obtained by fitting the masses of the nucleon and delta resonance to the energies of a spinning $B = 1$ Skyrmion. This approach results in a value

of F_π that is 124 MeV and therefore a little less than the experimental result. The second value of g is calculated in a later section by setting F_π to the experimental value and fitting the energy of the $B = 4$ Skyrmion to the He^4 mass.

Only static fields are considered in this paper, therefore the spatial components of the topological current vanish $B^i = 0$. As the topological current provides the source term for the field ω_μ then its spatial components can be set to zero, $\omega_i = 0$. For ease of notation, in the following we drop the subscript on the temporal component and write $\omega \equiv \omega_0$.

Taking into account the above comments, the static energy associated with the Lagrangian density (2.4) is given by

$$E = \int \left(\frac{1}{16} \text{Tr}(\partial_i U \partial_i U^\dagger) + \frac{M^2}{8} \text{Tr}(1 - U) - \frac{1}{2} \partial_i \omega \partial_i \omega - \frac{1}{2} \omega^2 - g \omega B^0 \right) d^3 x. \quad (2.5)$$

Note the negative signs associated with the energy in the ω field, as it is a temporal component.

Variation of the above energy with respect to ω yields the field equation

$$\partial_i \partial_i \omega - \omega = g B^0, \quad (2.6)$$

which is a linear equation for ω with a source term proportional to the topological charge density. The boundary condition is that ω vanishes at spatial infinity.

The problem at hand is therefore one of constrained energy minimization. The Skyrme field U is obtained as the field configuration, with a given topological charge B , that minimizes the energy (2.5), with ω determined uniquely by U as the solution of equation (2.6). In the following section an approximation technique is described and implemented to solve this problem.

Using equation (2.6), together with an integration by parts, the energy (2.5) can also be written as

$$E = \int \left(\frac{1}{16} \text{Tr}(\partial_i U \partial_i U^\dagger) + \frac{M^2}{8} \text{Tr}(1 - U) - \frac{1}{2} g \omega B^0 \right) d^3 x, \quad (2.7)$$

which will be useful later. Alternatively, the same procedure can be used to remove the last term in expression (2.5) in exchange for changing the two minus signs to plus signs in front of the ω dependent contributions. This form makes it transparent that the additional contribution to the energy from the inclusion of the ω field is non-negative.

3 Constructing multi-Skyrmions

Extensive numerical computations of the full nonlinear field equations of the standard Skyrme model, with massless pions, have determined the minimal energy Skyrmions with baryon numbers up to $B = 22$ [14]. These numerical results can be reproduced with a surprising accuracy using an approximation employing rational maps between Riemann spheres, known as the rational map ansatz [12]. This ansatz is briefly reviewed below.

In terms of the standard spherical polar coordinates r, θ, ϕ , introduce the Riemann sphere coordinate $z = e^{i\phi} \tan(\theta/2)$, given by stereographic projection of the two-sphere. Let $R(z)$ be a degree B rational map between Riemann spheres, that is, $R = p/q$ where p and q are polynomials in z such that $\max[\deg(p), \deg(q)] = B$, and p and q have no common factors. Given a rational map $R(z)$ the ansatz for the Skyrme field is

$$U(r, z) = \exp \left[\frac{if(r)}{1 + |R|^2} \begin{pmatrix} 1 - |R|^2 & 2\bar{R} \\ 2R & |R|^2 - 1 \end{pmatrix} \right], \quad (3.1)$$

where $f(r)$ is a real profile function which satisfies the boundary conditions $f(0) = \pi$ and $f(\infty) = 0$.

Applying the ansatz (3.1) to the baryon density (2.2) produces the expression

$$B^0 = -\frac{f'}{2\pi^2} \left(\frac{\sin f}{r} \frac{1 + |z|^2}{1 + |R|^2} \left| \frac{dR}{dz} \right| \right)^2, \quad (3.2)$$

from which it is simple to verify that the topological charge is indeed B , as a consequence of the boundary conditions on $f(r)$ and the fact that $R(z)$ has degree B .

For the particular choice $R = z$, the ansatz (3.1) reduces to the standard hedgehog ansatz for a spherically symmetric $B = 1$ Skymion. For $B > 1$ this ansatz does not provide any exact solutions, but for suitable choices of rational maps and profile functions, it provides excellent approximations to the true multi-Skymion solutions of the standard Skyrme model. In particular, the ansatz reproduces the correct symmetries of the true solutions, and the energy of the ansatz typically only overestimates the true energy of a multi-Skymion by the order of a percent.

Substituting the ansatz (3.1) into the energy of the standard Skyrme model leads to only one contribution which is sensitive to the properties of the rational map beyond its degree. This contribution originates entirely from the Skyrme term and involves the coefficient

$$\mathcal{I} = \frac{1}{4\pi} \int \left(\frac{1 + |z|^2}{1 + |R|^2} \left| \frac{dR}{dz} \right| \right)^4 \frac{2i \, dz d\bar{z}}{(1 + |z|^2)^2}. \quad (3.3)$$

Thus, within the rational map ansatz, the problem of finding the minimal energy Skymion in the standard Skyrme model reduces to the simpler problem of calculating the rational map which minimizes the function \mathcal{I} . Given the minimizing rational map, or more precisely the associated value of \mathcal{I} , the profile function can then be determined by minimizing an energy functional for $f(r)$.

The \mathcal{I} minimizing rational map for $B = 1$ is the spherically symmetric map $R = z$ and for $B = 2$ it is the axially symmetric map $R = z^2$, in agreement with the fact that the minimal energy $B = 2$ Skymion is axially symmetric [15, 16, 17].

For any $B > 1$ the map $R = z^B$ is axially symmetric, but it is not the \mathcal{I} minimizing map for $B > 2$. For $B = 3$ and $B = 4$ the \mathcal{I} minimizing maps are the unique maps with tetrahedral and cubic symmetry, respectively, and are given by [12]

$$R = \frac{z^3 - \sqrt{3}iz}{\sqrt{3}iz^2 - 1}, \quad R = \frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1}. \quad (3.4)$$

These symmetries again agree with those of the numerically computed Skyrmions in the standard Skyrme model [18].

In the vector meson model there is no Skyrme term and hence it appears that the coefficient \mathcal{I} is not relevant. However, notice that if the Laplacian term is neglected in the ω field equation (2.6) then ω is simply equal to a negative constant times the baryon density. Substituting this approximation into the energy (2.7) gives that the interaction term is a positive constant times the square of the baryon density. With the rational map expression for the baryon density (3.2) the angular part of this energy is precisely \mathcal{I} , and therefore this term arises in the vector meson model as a leading order contribution in a derivative expansion. Alternatively, a formal solution for ω can be written in terms of the Green's function for the massive Klein-Gordon equation. If this representation is applied to the interaction term in the energy then it becomes a non-local expression involving two factors of the baryon density. The angular part of this is therefore a non-local version of \mathcal{I} , which reproduces \mathcal{I} exactly in the limit in which the Green's function is replaced by a delta function.

The above arguments suggest that the \mathcal{I} -minimizing maps are also the appropriate maps for the vector meson model, and indeed it will be shown that the maps (3.4) yield low energy bound states, in contrast to the maps $R = z^3$ and $R = z^4$, for example. This is strong evidence that in the vector meson model the minimal energy Skyrmions for $B = 3$ and $B = 4$ have the same Platonic symmetries as in the standard Skyrme model. Additional evidence is provided by studying a (2+1)-dimensional analogue of this problem, where numerical solutions of the exact static field equations reveal an amazing similarity between solitons in the Baby Skyrme model and its vector meson version [19].

The approach applied below involves exploiting the symmetry of the Skyrme field to solve the field equation for ω by using an expansion in terms of symmetry adapted spherical harmonics. This aspect is reviewed in the following.

Given a subgroup $G \subset SO(3)$ of spatial rotations, introduce the symmetric harmonics $K_l(\theta, \phi)$, as a set of real orthonormal functions, each of which is a linear combination of spherical harmonics $Y_{lm}(\theta, \phi)$, that is,

$$K_l = \sum_{m=-l}^l \alpha_{lm} Y_{lm}, \quad (3.5)$$

where the coefficients α_{lm} are chosen so that each function K_l is invariant under the group G . Strictly speaking, the functions K_l should carry an additional index, since there may be more than one symmetric harmonic with a given value of l . However, for the specific calculations and symmetries discussed below, only the values $0 \leq l \leq 10$ will be required, and there is a unique symmetric harmonic for each l in this range; therefore for simplicity the additional index will be suppressed.

As an example, if $G = SO(2)$, then the axially symmetric harmonics are simply $K_l = Y_{l0}$.

The angular part of the rational map generated baryon density (3.2) is given by

$$b(\theta, \phi) = \frac{1}{4\pi} \left(\frac{1 + |z|^2}{1 + |R|^2} \left| \frac{dR}{dz} \right| \right)^2, \quad (3.6)$$

where the normalization is such that the integral of b over the two-sphere is equal to B .

For a G -symmetric rational map the angular baryon density b can be expanded in terms of symmetric harmonics as

$$b = \sum_l b_l K_l, \quad (3.7)$$

where $b_0 = B/(2\sqrt{\pi})$ from the chosen normalization. Similarly, ω can also be expanded in terms of symmetric harmonics as

$$\omega = \sum_l h_l(r) K_l, \quad (3.8)$$

where $h_l(r)$ are real profile functions which depend only on the radial coordinate r . The boundary conditions are that $h_l(\infty) = 0$ and $h_l(0) = 0$ for $l > 0$ with $h'_0(0) = 0$.

The baryon density expression (3.2) together with the expansions (3.7) and (3.8) reduces the ω field equation (2.6) to a set of profile function equations

$$h_l'' + \frac{2}{r} h_l' - \left(1 + \frac{l(l+1)}{r^2}\right) h_l' = -\frac{2g}{\pi} \frac{f' \sin^2 f}{r^2} b_l. \quad (3.9)$$

Substituting the rational map ansatz (3.1) and the expansions (3.7) and (3.8) into the energy (2.5) and performing the angular integration results in

$$E = E_\omega + \frac{\pi}{2} \int_0^\infty \{r^2 f'^2 + 2B \sin^2 f + 2M^2 r^2 (1 - \cos f) + \frac{4g}{\pi^2} f' \sin^2 f \sum_l b_l h_l\} dr, \quad (3.10)$$

where E_ω denotes a contribution to the energy which has no explicit f dependence.

The equation for f obtained from the variation of the energy (3.10) is

$$f'' + \frac{2}{r} f' - \frac{B}{r^2} \sin 2f - M^2 \sin f + \frac{2g}{\pi^2} \frac{\sin^2 f}{r^2} \sum_l b_l h_l' = 0. \quad (3.11)$$

For functions h_l and f which satisfy equations (3.9) and (3.11) the energy expression (2.7) can be used to rewrite the energy as

$$E = \frac{\pi}{2} \int_0^\infty \{r^2 f'^2 + 2B \sin^2 f + 2M^2 r^2 (1 - \cos f) + \frac{2g}{\pi^2} f' \sin^2 f \sum_l b_l h_l\} dr, \quad (3.12)$$

which is similar to (3.10) except that there is no longer a term which is independent of f and the coefficient in the final term has been halved.

Given a G -symmetric rational map $R(z)$ the angular baryon density (3.6) can be calculated and the coefficients b_l in the expansion (3.7) computed. The profile functions h_l and f can then be found by numerically solving the equations (3.9) and (3.11), using a heat flow algorithm. Finally, these profile functions are used to determine the energy from the expression (3.12). As mentioned earlier, it is found that truncating the expansions at $l = 10$ is sufficient to produce results of the required accuracy.

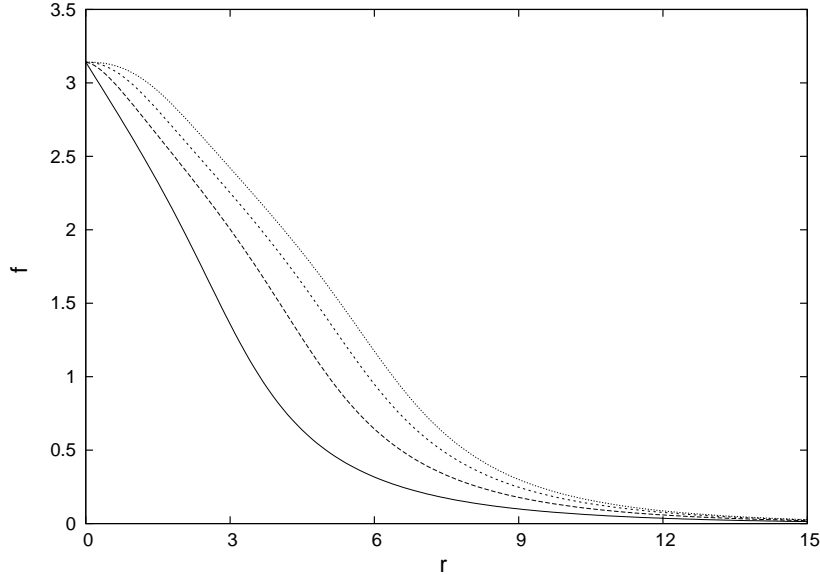


Figure 1: Profile functions $f(r)$ for $B = 1, 2, 3, 4$. The curves shift to the right with increasing B .

The simplest example is the spherical $B = 1$ Skyrmion with rational map $R = z$. There is only one spherically symmetric harmonic $K_0 = Y_{00} = 1/(2\sqrt{\pi})$ and hence only the profile functions f and h_0 need to be calculated. In this case the rational map ansatz is exact and the above procedure reproduces the earlier result [5]. For $g = 96.7$ the energy is found to be $E_1 = 44.04$ and the associated profile function f is plotted as the solid curve in Figure 1. The spherically symmetric field $\omega = h_0 K_0$ is presented as the solid curve in Figure 2.

Next consider the axially symmetric $B = 2$ map $R = z^2$. As mentioned earlier, the axially symmetric harmonics are $K_l = Y_{l0}$. However, the baryon density has an additional symmetry under rotations by 180° around an axis orthogonal to the $SO(2)$ symmetry axis, which implies that only symmetric harmonics with even l are needed. A computation of the angular baryon density produces the expansion coefficients

$$b = 0.564K_0 - 0.363K_2 + 0.107K_4 - 0.026K_6 + 0.006K_8 - 0.001K_{10} + \dots \quad (3.13)$$

A numerical solution of the profile function equations (for $l \leq 10$) results in the energy $E_2 = 87.44 < 88.08 = 2E_1$, and hence this is a state which is bound against the break-up into two single Skyrmions. Note that the binding energy is quite small, being less than 1%. However, the rational map ansatz is exact for a single Skyrmion and is only an approximation for multi-Skyrmions, therefore any computation of binding energies which compares to single Skyrmions will be a slight underestimate.

The profile function $f(r)$ is displayed in Figure 1 (the curves shift to the right with increasing B). Although the ω field now has a non-trivial angular dependence, it is useful

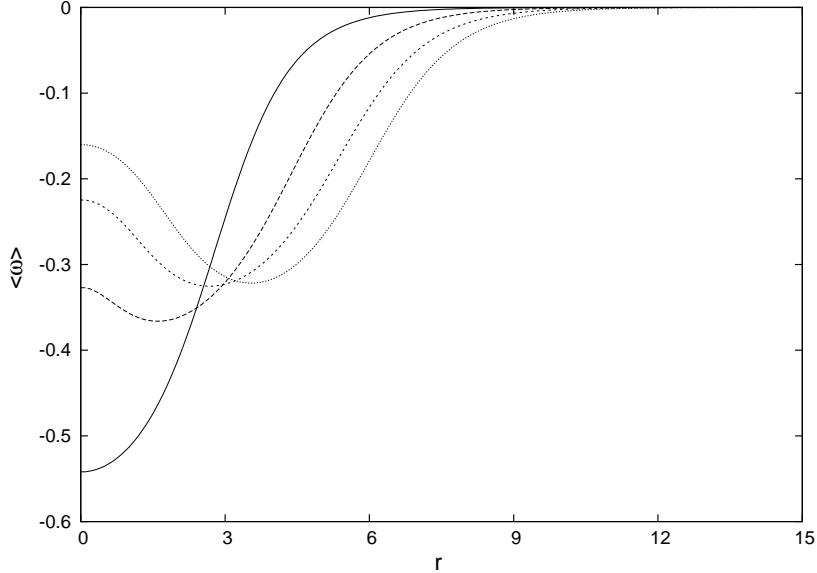


Figure 2: The spherical average $\langle \omega \rangle$ as a function of the radius r , for $B = 1, 2, 3, 4$. The curves shift up and to the right with increasing B .

to plot the spherically averaged value

$$\langle \omega \rangle = \frac{1}{4\pi} \int \omega \sin \theta d\theta d\phi, \quad (3.14)$$

which is displayed in Figure 2 (the curves shift up and to the right with increasing B). This reveals that although the ω field is again large and negative at the origin it takes its most negative value at a positive radius. This is to be expected because the source term for ω is the baryon density and this has a toroidal distribution. An isosurface plot of ω is displayed as the top right image in Figure 3, and the toroidal distribution is evident. Although isosurface plots of ω resemble the associated baryon density surfaces there are some differences, for example, the baryon density vanishes at the origin for $B > 1$ whereas, as seen above, the ω field at the origin is substantial.

Axially symmetric fields for larger baryon numbers can be studied by applying the same procedure as above to the rational map $R = z^B$. However, it is found that all axially symmetric fields of this type are unbound against the break-up into B single Skyrmions. For example, for $B = 3$ the energy is calculated to be $E_3^{\text{axial}} = 138.24 > 132.12 = 3E_1$. Of course, since the rational map approximation has been used to compute this energy it is expected that the true energy of the axial $B = 3$ field is slightly less than the above value, but the correction is unlikely to be large enough to yield a bound state. This is strong evidence that the $B = 3$ Skyrmion is not axially symmetric, in agreement with the usual Skyrme model. As mentioned earlier, with a Skyrme term and no vector mesons the minimal energy Skyrmion with $B = 3$ has tetrahedral symmetry and is described by the first rational map in (3.4). In the following it is shown that in the ω meson theory a tetrahedral bound state

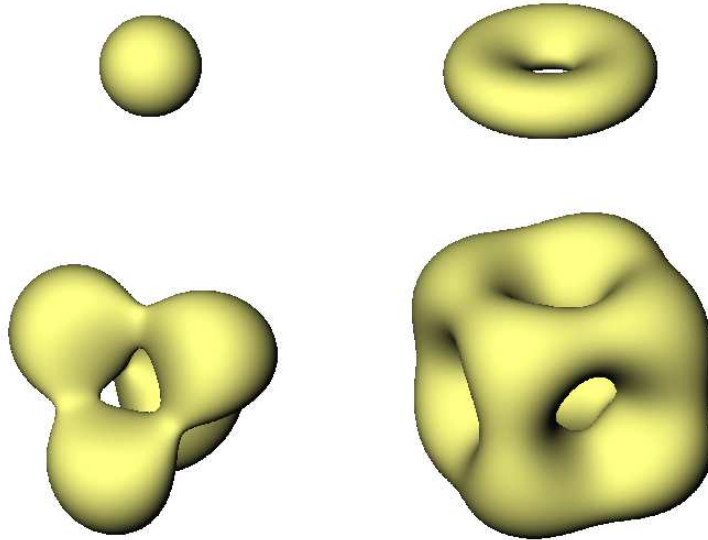


Figure 3: Isosurfaces of ω for $B = 1, 2, 3, 4$ (to scale). The isosurface values are given by $B = 1, \omega = -0.44$; $B = 2, \omega = -0.40$; $B = 3, \omega = -0.30$; $B = 4, \omega = -0.28$.

also exists with baryon number three, which suggests that it is the minimal energy $B = 3$ Skymion.

To study Platonic Skymions the symmetric harmonics K_l are required, where G is one of the Platonic symmetry groups. The construction of Platonic harmonics using invariant generating polynomials was introduced by Bethe and collaborators [20, 21], and this is briefly reviewed below for the tetrahedral case.

The ring of tetrahedrally invariant homogeneous polynomials is generated by three polynomials of degrees two, three and four,

$$p_2 = x_1^2 + x_2^2 + x_3^2, \quad p_3 = x_1 x_2 x_3, \quad p_4 = x_1^4 + x_2^4 + x_3^4. \quad (3.15)$$

To obtain a tetrahedral harmonic K_l , the first step is to determine the most general degree l homogeneous polynomial, k_l , that can be constructed as a linear combination of products of the polynomials (3.15). The coefficients in the polynomial k_l are then fixed (up to an irrelevant overall factor) by requiring that k_l solves Laplace's equation.

Note that for $l = 2, 3, 5$ there is only one possible contribution to k_l , namely p_2, p_3 and $p_2 p_3$ respectively. Of these three polynomials only p_3 satisfies Laplace's equation, hence there are no tetrahedral harmonics with $l = 2$ or $l = 5$, and obviously none with $l = 1$.

Given the tetrahedrally invariant polynomial k_l then k_l/r^l is a function of only the angular coordinates θ, ϕ and, with a suitable normalization, is the required tetrahedral harmonic K_l , which can be decomposed into spherical harmonics Y_{lm} . In addition to the above construction, Platonic harmonics may also be determined using a projector technique [22], but in either

case the results are

$$\begin{aligned}
K_0 &= Y_{00}, \quad K_3 = \frac{i}{\sqrt{2}}(\bar{Y}_{32} - Y_{3-2}), \quad K_4 = \frac{1}{2}\sqrt{\frac{5}{6}} \left(Y_{44} + Y_{4-4} + \sqrt{\frac{14}{5}}Y_{40} \right), \\
K_6 &= \frac{\sqrt{7}}{4} \left(Y_{64} + Y_{6-4} - \sqrt{\frac{2}{7}}Y_{60} \right), \quad K_7 = i\frac{\sqrt{33}}{12} \left(\bar{Y}_{76} - Y_{7-6} + \sqrt{\frac{13}{11}}(\bar{Y}_{72} - Y_{7-2}) \right), \\
K_8 &= \frac{1}{24}\sqrt{\frac{195}{2}} \left(Y_{88} + Y_{8-8} + 2\sqrt{\frac{7}{65}}(Y_{84} + Y_{8-4}) + 3\sqrt{\frac{22}{65}}Y_{80} \right), \\
K_9 &= \frac{i}{4}\sqrt{\frac{13}{2}} \left(\bar{Y}_{96} - Y_{9-6} - \sqrt{\frac{3}{13}}(\bar{Y}_{92} - Y_{9-2}) \right), \\
K_{10} &= \frac{\sqrt{561}}{48} \left(Y_{10,8} + Y_{10,-8} + \frac{6}{\sqrt{51}}(Y_{10,4} + Y_{10,-4}) - \sqrt{\frac{130}{187}}Y_{10,0} \right).
\end{aligned} \tag{3.16}$$

Substituting the first rational map in (3.4) into the angular baryon density (3.6) and expanding in terms of tetrahedral harmonics produces

$$b = 0.846K_0 - 0.585K_3 - 0.070K_4 - 0.127K_6 + 0.026K_7 + 0.002K_8 + 0.022K_9 + 0.006K_{10} + \dots \tag{3.17}$$

Using the above coefficients the associated profile functions can be computed (f is presented in Figure 1 and the spherical average $\langle \omega \rangle$ is plotted in Figure 2) and the energy found to be $E_3 = 126.63 < 131.48 = E_2 + E_1$, and therefore a bound state. A tetrahedrally symmetric ω isosurface is displayed as the lower left image in Figure 3.

Turning attention to $B = 4$, the required rational map is the second map in (3.4) and has cubic symmetry. The cubically symmetric harmonics are, of course, a subset of those with tetrahedral symmetry. The tetrahedral polynomials p_2 and p_4 in (3.15) are also invariant under the cubic group, but p_3 is not. However, $p_6 \equiv p_3^2$ is invariant under the cubic group, and indeed p_2, p_4, p_6 are the generating polynomials. This implies that the required cubic harmonics are the tetrahedral harmonics K_l (3.16) with even l .

The expansion coefficients of the $B = 4$ angular baryon density are given by

$$b = 1.128K_0 - 0.601K_4 - 0.050K_6 + 0.076K_8 + 0.009K_{10} + \dots \tag{3.18}$$

and the results of computing the profile functions are again displayed in Figure 1 and Figure 2. The energy is calculated to be $E_4 = 159.67$, and this confirms that this configuration is bound against the break-up into all possible lower charge clusters. A cubically symmetric ω isosurface is displayed as the lower right image in Figure 3.

The energy results for this coupling, which recall is $g = 96.7$, are summarized in the second and third columns of Table 1, where the energy is also presented as a ratio to that of a single Skyrmion (this is convenient for the bound state comparison).

A range of values for the coupling g have been investigated and the results are qualitatively similar. As discussed earlier, the value $g = 96.7$ is taken from [5], which involves

	$g = 96.7$		$g = 34.7$			
B	E_B	E_B/E_1	E_B	E_B/E_1	E_B in MeV	Experiment
1	44.04	1.000	22.53	1.000	996	939
2	87.44	1.985	45.20	2.006	1999	1876
3	126.63	2.875	65.88	2.924	2913	2809
4	159.67	3.626	84.28	3.741	3727	3727

Table 1: The energy, E_B , of the charge B Skyrmion and the ratio E_B/E_1 , of the energy to that of a single Skyrmion, for two different values of the coupling g . For the second value of the coupling the energy is also given in MeV and the corresponding experimental value is listed for comparison.

fitting properties of the single Skyrmion to the nucleon and delta, whilst treating F_π as a free parameter. However, it is known that this method of parameter fitting can be problematic, in any Skyrme model, because of difficulties associated with the rigid rotor approximation [23]. Access to the properties of multi-Skyrmions allows the following alternative procedure to be applied.

All the physical parameters, meson masses m_π , m_ω , and the pion decay constant F_π , are set to the experimental values listed earlier, leaving only the single parameter g to be determined. A sensible way to fit this parameter is to match the energy of the $B = 4$ Skyrmion to the He^4 mass, which is 3727 MeV. The reason this is a good approach is that the ground state of He^4 has zero spin and isospin, therefore there are no issues to address regarding the inclusion of quantum energies associated with spin and isospin. Adopting this approach yields the value $g = 34.7$ and the associated Skyrmion energies, plus the experimental masses for comparison, are listed in columns four to seven of Table 1.

One point to note is that reducing g leads to a reduction in the relative binding energies. In particular, the data presented in Table 1 reveals that for this new value of the coupling $E_2 = 2E_1 \times 1.003$, and therefore it appears that the $B = 2$ Skyrmion is not a bound state. However, recall that the rational map ansatz is only an approximation for $B > 1$ and the true energy is expected to be anything up to the order of a percent lower than the approximate value. As the value of E_2 is so close to that of $2E_1$ then this correction is expected to produce a $B = 2$ bound state, with a small binding energy. In fact, experimentally the deuteron binding energy is around 2 MeV, which is only about 0.1% of the deuteron mass. Therefore, if a reasonable comparison with experimental data is to be achieved, then even a tiny overestimate of the $B = 2$ energy would indeed be expected to result in an apparently unbound state.

A comparison of the last two columns in Table 1 shows that all the calculated energies are within 7% of the experimental values, which is very reasonable given that the theory has only a single free parameter g . This accuracy is comparable to that in the standard Skyrme model, when the pion decay constant is treated as a free parameter whose value emerges as considerably lower than the experimental one.

The Skyrmion energies presented are the classical energies and do not include quantum

contributions associated with spin and isospin. These quantum energies are traditionally added via a zero mode quantization involving the calculation of inertia tensors from the classical solution [24]. Expressions for the required inertia tensors have been calculated in terms of rational maps and profile functions [25, 26] and therefore the results in this paper could be used to calculate these quantum corrections to the presented classical energies.

4 Conclusion

Multi-Skyrmions have been investigated in a theory without a Skyrme term but including the ω vector meson to provide a scale for the Skyrmion. The approach employed here involved using the rational map ansatz and therefore an obvious avenue for future research is to perform full field simulations of this theory, to test the accuracy of the approximations used. In particular, in the standard Skyrme model it is known that for baryon numbers above seven, there is an important qualitative difference between multi-Skyrmions in a theory with massive or massless pions [27, 28]. It would be of interest to know if a similar result holds for the theory with vector mesons and no Skyrme term.

Extending the current analysis to a theory including other vector mesons, starting with the ρ meson along the lines of [6], is clearly of importance. Furthermore, the emergence of a Skyrme model with vector mesons from AdS/QCD [13] suggests that a Skyrme term should also be included, even if it is not required to provide a scale for the Skyrmion. The results presented in this paper suggest that the vector meson and Skyrme term theories have very similar properties, and therefore one might expect that a theory which includes both contributions will also have similar universal features.

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